Further Comments on Realization of Riemann Hypothesis via Coupling Constant Spectrum

R. Acharya
Department of Physics
Arizona State University
Tempe, AZ 85287-1504

May 7, 2009

Abstract

We invoke **Carlson's theorem** to justify **and** to confirm the results previously obtained on the validity of Riemann Hypothesis via the coupling constant spectrum of the zero energy S-wave Jost function a la N. N. Khuri, for the real, repulsive inverse-square potential in non-relativistic quantum mechanics in 3 dimensions.

In a previous note (hereinafter referred to as I) [1], we have exhibited the solution for the S-wave Jost function at zero energy (i.e., $k^2 = 0$) for the real, repulsive inverse-square potential, $V(r) = \frac{\lambda}{r^2}$ $(\lambda > 0)$. We then argued that it is justified to identify the S-wave Jost function, $\mathcal{X}(s) \equiv F_+((s(s-1); k^2 = 0)$ with the Riemann $\xi(s)$ function [2] (up to an entire function of order one, with **No** zeros) where the coupling constant λ is **constrained** to satisfy the relation, i.e.,

$$\lambda = s(s-1) \tag{1}$$

The argument hinged on demonstrating that the S-wave Jost function has **only** zeros on the line $s_n = \frac{1}{2} + i\gamma_n'$, $n = 1, 2, 3, \ldots \infty$. This was accomplished by first demonstrating that the analogue of Eq. (1.2) of Khuri's paper [3] (**without** the presence of the potential $V(r)$ under the sign of integration), i.e.,

$$\left[Im\lambda_n(i\tau)\right] \int_0^\infty \left|f(\lambda_n(i\tau);\ i\tau;\ r\right|^2 dr = 0 \tag{2}$$

holds, where the all-important constraint, i.e.,

$$\int_0^\infty \left|f(\lambda_n(i\tau);\; i\tau;\; r\right|^2 dr < \infty \tag{3}$$

is satisfied, i.e.,

$$\left[Im\lambda_n(i\tau)\right]\frac{1}{\tau}\frac{2}{\pi}\int_0^\infty rK^2_{\nu(i\tau)}(r)dr = 0 \tag{4}$$

where

$$\int_0^\infty rK^2_{\nu(i\tau)}(r)dr = \frac{1}{8}\frac{\pi\nu(i\tau)}{\sin\pi\nu(i\tau)} \tag{5}$$

with $\nu(i\tau) = \sqrt{\lambda(i\tau) + \frac{1}{4}}$, $\nu(i\tau) \neq 1, 2, 3, \ldots\infty$. Then, Eq. (3) follows for $\tau \neq 0$ $(\tau > 0)$.

Thus,

$$Im\lambda_n(i\tau) = 0 \tag{6}$$

We then followed Khuri and arrived at the conclusion,

$$\lambda_n(0) \equiv s_n(s_n - 1) \tag{7}$$

for **all** $\boldsymbol{n}$, is **real** and **negative**. When $\lambda_n(0)$, for all n is real and negative, (and by rescaling such that $\lambda_n < -\frac{1}{4}$, $\nu(0) = \sqrt{\lambda_n(0) + \frac{1}{4}}$ becomes imaginary and hence R.H.S. of Eq. (5) will become the ratio of $\frac{\pi\nu(0)}{\sinh\pi\nu(0)}$, at $\tau = 0$. In other words, the zero energy coupling spectrum lies on the negative real line for $V(r) = \lambda V^*(r)$, $V(r) = \frac{1}{r^2}$, $V^* \geqslant 0$. In view of Eq. (1), i.e., $\lambda = s(s-1)$, one then concluded that $\mathcal{X}(s)$ has all its infinite number of nonzero zeros on the "critical" line, Re $s_n = \frac{1}{2}$.

The **key question** arises whether or not, the positions γ_n of the infinite number of non-zero zeros on the critical line, **coincide** with the Riemann Hypothesis, i.e.,

$$s_n = \frac{1}{2} \pm i\gamma_n,\; n = 1, 2, 3\ldots\infty$$

(Recall that Hardy has proved that $\xi(s)$ has **infinite number** of zeros on the critical line.)

In other words, one **must** demonstrate that the positions of the zeros obtained from the potential, $V(r) = \frac{\lambda}{r^2}$, i.e.

$$s_n^{'} = \frac{1}{2} \pm i\gamma_n^{'},\; n = 1, 2, 3\ldots\infty$$

coincide exactly with the positions (γ_n) of the Riemann zeros. This was **not** done in our earlier note! **We propose to demonstrate this here.** Our tool is Carlson's theorem [4].

Recall,

$$\mathcal{X}(s^{'}) = e^{\pi s^{'}}. \tag{8}$$

and

$$\xi(s^{'}) = Ce^{A(s^{'}+1)} \tag{9}$$

for real $s^{'} \geqslant 0$, $s^{'} = s - 1$, where we are stating the **established** fact that **both** $\mathcal{X}(s)$ and $\xi(s)$ are **entire** with **No zeros** for real $s^{'} > 0$.

Then, we invoke Hadamard's factorization theorem to set

$$e^{a+\frac{bs^{'}}{m}}\mathcal{X}(\frac{s^{'}}{m}) - \xi(\frac{s^{'}}{m}) = 0, \quad \frac{s^{'}}{m} = 1, 2, 3 \ldots \tag{10}$$

Eq. (10) is satisfied iff

$$Ce^{A} = e^{a} \text{ and } A = b + \pi \tag{11}$$

We now **invoke Carlson's theorem** [4]:
In order that each entire function $f(z)$ satisfying the conditions

$$f(z) = 0(1)e^{\alpha|z|}, \text{ for some } \alpha < \infty \tag{12}$$

$$f(iy) = 0(1)e^{\beta|y|}, \text{ for some } \beta < \pi \tag{13}$$

$$f(n) = 0 \text{ for each positive integer ‘}n\text{’} \tag{14}$$

$$\text{then } f(z) \equiv 0 \text{ !} \tag{15}$$

Thus, we conclude that

$$e^{a+\frac{bs^{'}}{m}}\mathcal{X}\left(\frac{s^{'}}{m}\right) = \xi\left(\frac{s^{'}}{m}\right), \textbf{ all } s^{'} \tag{16}$$

Clearly Eq. (10) is satisfied for **all** $\frac{s^{'}}{m} > 0$, when Eq. (11) holds. An analytic (entire) function which vanishes in a subdomain of its region of analyticity, **must** necessarily vanish in its “entire” domain of analyticity, i.e., for **all** $s^{'}$! Thus, in a sense, one need not invoke Carlson's theorem to arrive at Eq. (16)!

It is now obvious that the zeros on the critical line of the zero energy Jost function $\mathcal{X}(s)$ and the zeros (infinite number, established by G. H. Hardy) of Riemann's $\xi(s)$ function must necessarily coincide, i.e.,

$$s_n^{'} = \frac{1}{2} \pm i\gamma_n^{'} = s_n = \frac{1}{2} \pm i\gamma_n, \quad n = 1, 2, \ldots \infty.$$

We can explicitly verify this assertion as follows:

We invoke the Hadamard factorization for every entire function $g(z)$ of order one and "infinite type" (which guarantees the existance of **infinitely many** non-zero zeros [valid for **all** z!] [4]:

$$g(z) = z^m e^B e^{Dz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n}\right) \tag{17}$$

where 'm' is the multiplicity of the zeros (so that $m = 0$ for a simple zero). We can now apply Eq. (17) to the zero energy Jost function $\mathcal{X}(s^{'})$ and the Riemann's $\xi(s^{'})$ function, assuming, $s_n^{'} = \frac{1}{2} + i\gamma_n^{'}$ for $\mathcal{X}(s^{'})$ and $s_n^{'} = \frac{1}{2} + i\gamma_n$ for Riemann's $\xi(s^{'})$.

This leads to the equality [Eq. (16)]:

$$\begin{aligned} &e^H e^{L\frac{s^{'}}{m}} \prod_{n=1}^{\infty} \left(1 - \frac{s^{'}}{ms_n^{'}}\right) \exp\left(\frac{s^{'}}{ms_n^{'}}\right) \\ &\quad = e^J e^{K\frac{s^{'}}{m}} \prod_{n=1}^{\infty} \left(1 - \frac{s^{'}}{ms_n}\right) \exp\left(\frac{s^{'}}{ms_n}\right). \end{aligned} \tag{18}$$

First set $s^{'} = 0$ in Eq. (18):

$$\Longrightarrow \qquad H = J \tag{19}$$

Next set, $\boldsymbol{s^{'} = ms_n}$, $s_n = \frac{1}{2} + i\gamma_n$

$$\begin{aligned} \Longrightarrow \qquad &e^{Ls_n} \prod_{n=1}^{\infty} \left(1 - \frac{s_n}{s_n^{'}}\right) \exp\left(\frac{s_n}{s_n^{'}}\right) \\ &\quad = e^{Ks_n} \prod_{n=1}^{\infty} (1 - 1) \exp(1). \end{aligned} \tag{20}$$

Since RHS of Eq. (20) **vanishes**, we conclude that s_n must equal $s_n^{'}$:

$$\therefore \gamma_n = \gamma_n^{'} \tag{21}$$

This **establishes the key result** that we set out to prove: **The Riemann zeros on the critical line do coincide precisely with the zeros of the zero energy $\boldsymbol{S}$ wave Jost function for $\boldsymbol{V(r) = \frac{\lambda}{r^2}}$ $\boldsymbol{(\lambda > 0)}$** [5].

References and Footnotes:

[1] R. Acharya: Realization of the Riemann Hypothesis via Coupling Constant Spectrum, arXIV: math-ph/0803.1818 v5

[2] The Riemann Hypothesis: **A Resource for the Afficinado and Virtuoso Alike**. Borwein, Choi, Rooney and Weirathmueller (Editors), Springer (2008);
The theory of the Riemann Zeta-Function, Second Edition, E.C. Titchmarsh, revised by D. R. Heath-Brown, Clarendon Press, Oxford (1986).

[3] N. N. Khuri: Inverse Scattering, the Coupling Constant Spectrum and the Riemann Hypothesis, arXiv: hep-th/0111067 v 1, 7 November 2001;
K. Chadan and P. C. Sabatier: Inverse Problems in Quantum Scattering Theory, Second Edition, Springer-Verlag (1989);
R. G. Newton: Inverse Schrödinger Scattering in 3 dimensions, Springer-Verlag (1989).

[4] B. J. Levin: Distribution of Zeros of Entire Functions, American Math. Soc. Traslations of Math monographs, Vol. 5, Providence, RI, 1964.
See also, B. Ghusayni, Entire Functions of Order One and Infinite Type (preprint).

[5] We hasten to correct misstatements in Reference 1. The sentence immediately following Eq. (30) must be deleted. Eq. (40) and Eq. (41) must be replaced by

$$\alpha = \ln 2, \tag{40}$$

$$\beta = 0 \tag{41}$$

One arrives at this by setting $\mathcal{X}(0) = \mathcal{X}(1) = 1$ and $\xi(0) = \xi(1) = \frac{1}{2}$.

Acknowledgement

I am grateful to Irina Long for her generous help in the preparation of this manuscript.